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THE LIMIT ANALYSIS AND DESIGN OF TENSION-FIELD BEAMS

by

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THE LIMIT ANALYSIS AND DESIGN OF TENSION-FIELD BEAMS*

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Abstract

The purpose of this paper is to develop theoretical analysis that may be applied to discuss the general problem of the behavior of tension-field beams. The analysis is based upon the application of the theory of limit analysis to a plastic-rigid model that approximates the actual behavior of such a beam. It is shown that Wagner's tension-field beam analysis is a special case of the present analysis. Apart from some brief remarks on the immediate steps required in the development of the investigation, no further discussion of the problem is attempted here.

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1. Introduction.

The general problem of the determination of the load-carrying capacity of tension-field beams is one of great importance in aircraft structural engineering. These members are primary components in the wings and fins of many types of aircraft, and also occur elsewhere in sharply-tapered form as frames in the hulls of flying boats. Much research, both from experimental and theoretical standpoints, has been directed towards the better understanding of this problem. The literature that describes this research is very extensive. In the following brief summary, the references given are intended, in the main, merely to be representative.

It should be noted first that a tension-field beam comprises two stiff flanges which are usually parallel or nearly so, a thin web, and a number of stiffeners which are usually evenly-spaced and positioned at right angles to the flanges. Such a beam, regarded as a cantilever with a concentrated vertical shear load applied at its free end, is illustrated in Fig. 1. The primary function of the flanges is to carry compressive and tensile loads, and thereby to react the bending moment at a section. Again the main purpose of the web is to carry shear, and thereby to react the applied shear load. Lastly, although the stiffeners delay the buckling of the web, their most important function is to maintain the separation of the flanges. Up to the time that the web buckles, the beam may be correctly regarded as a conventional beam - with shear as well as bending

stiffness - and treated accordingly (see, for example, [1]).* However, after buckling of the web has occurred the problem assumes a radically different aspect. Experiments made on typical beam designs (for which the flange stiffness is relatively high) show that when the web buckles it develops a series of waves running roughly at 40° to the flanges, and that the overall shear stiffness of the beam drops at once to about three-quarters of its former value. Prior to the buckling of the web, the tensile and compressive stresses in the web are approximately equal; but after buckling of the web has occurred the tensile stress continues to increase roughly at the same rate as before whereas the compressive stress continues to increase only much more slowly. Subsequent to buckling of the web, the increase in applied shear load is carried almost entirely by the tensile stress in the web, but as before the bending moment at a section is still mainly reacted by compressive and tensile loads in the flanges. Other features appear with buckling of the web. Thus the flanges become subjected to distributed transverse loads from the web, and their tendency to approach one another under the action of these loads is resisted by the stiffeners in which compressive stresses are set up. Moreover these transverse loads tend to bow the flanges between the points of stiffener attachment, and accordingly secondary bending moments, which are

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*Numbers in square brackets refer to the bibliography given at the end of the paper.

greatest near the points of stiffener attachment, are induced in the flanges. The web would also act on the stiffeners at the ends of the beam in much the same way were it not for the fact that in practice the beam design is almost certain to embody special features near its ends and these will modify considerably the effects just described. The buckles tend to be smoothed out by the flanges and also, although to a lesser extent, by the stiffeners. As the shear load increases the amplitude of the buckles grows and their wavelength decreases. Ultimately failure of the beam must occur, and this can happen in any one, or a combination, of a number of ways depending upon the detailed design of the beam. For example, the web may tear away from the rivets that attach it to the flanges; the compression flange may fail by instability; or the stiffeners may fail by instability under the compression stresses induced in them by the web, and then follow the buckles in the web. In any design that is satisfactory from a weight point of view the various competitive types of collapse should occur together at failure. This condition is quite often approximated through empirical design modifications following tests. In any event it is important to note that buckling in the web is usually very highly-developed indeed prior to failure of the beam - the nominal shear stress in the web at failure may be as much as twenty times the buckling shear stress of the web, and approaching the ultimate stress.

The first attempt to treat the post buckling behavior of a beam with a thin shear web is due to Wagner [2] in his theory

of the completely-developed tension field. The immediate urgency of this problem was due to the progress in the design of aircraft, but the general phenomena which occurred had been noted much earlier by Stephenson [3] in experiments on railway bridge structures. Wagner's theory assumed that as soon as the web buckled it could only sustain tension, the direction of this tension being of course parallel to the main wave crests and troughs. Design experience soon showed that this hypothesis was too serious an over-simplication of the situation on which to base a really adequate analysis. In particular the compressive stress in the web could not be entirely neglected. An appreciation of this fact marked the next stage in the development of the theory, and gave rise to the idea of the incompletely-developed tension fixed. The new theory assumed that there was a compressive stress in the web acting in a direction perpendicular to the direction of the waves, and that its magnitude was independent of the amount of buckling. Experiments designed to check the theory soon showed however that the compressive stress in the web continued to increase, although much more slowly than did the tensile stress, with the development of buckling. The inherent difficulty of the problem now led to the development of improved semi-empirical theories (see [4] and [5]). Such theories have been used as a basis for minimum weight analyses (see [6]).

The task of developing a complete theory of tension-field beams through an analysis which treats the post-buckling behavior of the web according to the von Kármán equations (see [7])

for the large elastic deflections of thin plates has proved up to the present time to be too difficult a task. This is evidenced, for example, by the attempt of Leggett and Hopkins [8] to approximate such an analysis through an energy method of approach based upon certain simplifying assumptions. The theoretical results obtained by these authors were compared with experimental results by Crowther and Hopkins [9].

In the absence of a complete theory of the problem reliance is necessarily mainly to be made upon the approximate and empirical analyses based upon refinements of the analysis initiated by Wagner and upon empirical conclusions drawn from experimental observations. This is not to say that the present position is altogether unsatisfactory so far as conventional designs of tension-field beams are concerned. However the lack of a general theory does necessarily limit the understanding of the problem.

In the present paper an alternative approach is offered to the analysis of the general problem which it is believed will yield useful results particularly in the important field of minimum weight design. Although the present discussion of the application of the general theory developed here is with reference to beams with parallel flanges, this is not a necessary restriction (see [2]). The notable success of the methods of limit analysis and design, particularly in their application to barred- and framed-structures, has suggested and encouraged their use in the present study. A brief summary of the literature

available on limit analysis and design will now be given before we proceed to consider the analysis of the present problem in more detail.

The reader is referred to the books by Van den Broek [10] and by Prager and Hodge [11] for detailed accounts of the methods of limit analysis and design together with their various applications. Briefly limit analysis (or design) in structural engineering may be defined as a procedure in which the analysis (or design) of the structure is based upon the ultimate behavior of the structure. Prager [12] has defined limit analysis as being concerned with the estimation of the load intensity at which a given statically-indeterminate structure ceases to be serviceable; and limit design, on the other hand, as being concerned with the allocation of sufficient local yield strength to the various parts of the structure in such a manner that this structure remains serviceable under given conditions of loading. The basic concepts of limit design may be traced back to the introduction of ductile materials - wrought iron and mild steel - in engineering structures. However such concepts were only formalized some three decades ago. The earliest applications were to beams but later frames were also successfully treated. Following the work on extremum principles in plasticity by Drucker, Greenberg and Prager (see [11]) and by Hill [13] considerable, and continuing, stimulus has been given to applications to more complicated structural elements such as plates and shells (see [14]). As Hill [13] has pointed out the principles of limit analysis are most conveniently formulated for

a perfectly-plastic material that is rigid wherever the stress is below the yield limit.

The present study is usefully compared with the recent related work by Onat and Shield [15] and by Leth [16] who discuss the effect of shearing forces on the load-carrying capacity of wide beams and of I-beams, respectively. This comparison shows that similarities exist in respect of the adoption of the methods of limit analysis, but, on the other hand, dissimilarities exist in respect of the correspondence between the actual physical structure and the mathematical model due to the present need to simulate the shear buckling of the web.

The object of the present paper is to develop a theoretical analysis that may be applied to discuss the general problem of the behavior of tension-field beams. The theory is based upon the application of limit analysis to a plastic-rigid model that approximates the actual mechanical behavior of such a beam. The difficulties in the choice of a suitable mathematical model arise chiefly in two ways. First there is the need to account for buckling of the web and the relatively small contribution of the web compressive stress in reacting the applied shear load once buckling is well-developed. Second there is the need to account for strain-hardening effects in the light-alloy material of which the beam is normally expected to be made. In respect of the former difficulty it is of special interest that a somewhat analogous situation occurs in certain problems of soil mechanics and in the design of masonry structures where the

the material is unable to take appreciable tension so that applied loads must be reacted almost entirely through compressive loads set up in the material. The methods of limit analysis have been successfully applied to provide solutions of problems in these fields on the basis of the assumption that the material is unable to take any tension (see [17] and [18]). The assumption of zero tensile yield stress is not essential but its adoption does greatly simplify the analysis with comparatively little loss in the accuracy of the values of the limit loads. Accordingly, as originally proposed by Wagner [2], it will be supposed that the web is unable to carry any compressive stress. In respect of the latter difficulty previously mentioned, the effects due to strain-hardening may be taken into account through an approximate, although reasonably effective, manner by the choice of an effective yield stress somewhat higher than the yield point. The reader is referred to a recent paper by Dwight [19] for a discussion of a plastic design method for aluminum structures. It does not appear practicable to assess directly the error involved in making these simplifications in the actual mechanical behavior of the beam, and the justification of results founded upon the present analysis must ultimately be based upon experimental results.

Finally the reader should note that the relative simplicity of limit analysis is due to the fact that an elaborate and complicated elastic-plastic analysis is avoided. Furthermore even when exact values of the limit loads are difficult to

obtain, reasonably close upper and lower bounds to these loads may be found through the use of the limit design theorems of Drucker, Greenberg and Prager (see [11]). These theorems will not be used in the present paper, but in simple terms they mean that the structure withstands the applied loads through the optimum distribution of internal stresses and that the limit load is reached as soon as a mechanism of collapse is available.

The general theory that is developed here applies in slightly modified form also to the problem of the strength of composite wall structures that are composed of a mild steel skeleton reinforced by bricks. The modification is due of course to the fact that the brick cannot sustain appreciable tension although it will sustain compression. This application is not discussed further in the present paper.

2. Notation.

Part of the notation to be used is now described (see Fig. 2). Let the web be rectangular and be supposed vertical with one pair of opposite edges horizontal. Let the flanges be rectangular in cross-section, one principal axis of a cross-section being vertical and lying in the plane of the middle surface of the web. The stiffeners are supposed vertical, at constant pitch and symmetrically attached to the web. Let O(x,y) be a rectangular Cartesian frame of reference in the middle surface of the web, the x- and y-axes being horizontal and vertical, respectively. Let

ℓ,d,t = web length, depth and thickness;

$2a,b$ = depth and width of flange cross-section;

σ_o = tensile yield stress;

F,Q,M = axial and shear forces, and bending moment in flange;

u_F,v_F = velocity components in neutral axis of flange;

σ_x, σ_y, τ_{xy} = stresses in web averaged across thickness;

v_x,v_y = velocity components in web averaged across thickness; and

$\varepsilon_x,\varepsilon_y,\gamma_{xy}$ = strain components in web averaged across thickness.

Other notation is defined when first introduced.

3. Preliminary Analysis.

It is convenient, before proceeding to the detailed analysis of the flanges and the web, to describe the general approach which follows closely that developed by Prager (see [20] and [21]).

The mechanical behavior of a perfectly-plastic rigid continuum is specified in terms of independent generalized strain-rates e_i and stresses $\sigma_i (i = 1,2,\ldots,I)$, these quantities being associated in such a way that the rate of plastic work per unit volume is

$$W = \sigma_1 e_1 + \sigma_2 e_2 + \ldots. + \sigma_I e_I. \qquad (3.1)$$

The generalized strain-rates are derived from generalized velocities v_j $(j = 1,2,\ldots,J)$ which specify the type of plastic flow envisaged and reflect the basic kinematical assumptions of the theory. In general once the v_j's are assigned and the e_i's have been determined, the σ_i's are best found from considerations of

virtual work. The e_i's may always be modified in such a way to have the same dimensions as, for example, conventional strain-rates. The same applies to the σ_i's. Indeed if e_i is modified to e_i/c_i, where c_i is some constant, then σ_i must be modified to $c_i \sigma_i$, if W is to remain unchanged. This modification is essential if, as is often convenient, the e_i's and σ_i's are to be regarded as the rectangular Cartesian co-ordinates of points in I-dimensional strain-rate and stress spaces. Further the points e_i and σ_i , and hence the states of strain-rate and stress, may be associated with the vectors, say $\underline{e}$ and $\underline{\sigma}$ respectively, having their initial points at the origin and their terminal points at e_i and σ_i . Moreover it may be convenient to superimpose the strain-rate and stress spaces in which case all e_i's and σ_i's must be chosen so as to have the same dimensions. In the absence of effects such as strain-hardening, viscosity and inertia, no fundamental time is involved in the problem, and accordingly the velocities are only determinate to within an arbitrary constant factor of proportionality.

The physical laws involved in the problem are those of equilibrium; plastic flow without fracture or, within a rigid (elastic)region, strain compatibility; and mass conservation. These laws are just sufficient to provide the necessary set of mathematical relations on the unknown field quantities and either the external forces or the geometry according as the problem is one of analysis or design. The physical laws provide not only field equations and boundary conditions but also certain fundamental requirements in respect of continuity on field quantities.

The equations of equilibrium involve both internal and external stresses. Any internal stress which does not explicitly occur in W may be associated with a strain-rate that is identically zero. Such a stress is called a reaction. The actual derivation of the equations of equilibrium is standard.

The yield condition is expressed by

$$f(\sigma_1, \sigma_2, \ldots, \sigma_I) = 0 \qquad (3.2)$$

where the yield function f assumes negative values for stress states below the yield limit. In the simplest case f is everywhere continuously differentiable with respect to the σ_i's, i.e. the yield surface has everywhere a continuously-turning tangent plane, and f is called regular. Otherwise the yield surface is called singular, and points where all the $\frac{\partial f}{\partial \sigma_i}$'s do not exist are called singular points. Note that, although it is convenient to use geometrical language, expressions such as 'yield surface' must be interpreted appropriately in any particular context. The most important example of a singular yield surface occurs when the (closed) yield surface S comprises a finite number of (open) regular surfaces $S_r (r = 1,2,\ldots,R)$ so that singularities may occur only at the intersections of the S_r's. This situation arises when more than one physical mechanism is available to admit plastic flow at certain stress states. In such a case the yield limit Eq.(3.2) is expressed in terms of a finite number of

continuously differentiable yield functions

$$f_r(\sigma_1, \sigma_2, \ldots, \sigma_I) \quad (r = 1,2,\ldots,R) \tag{3.3}$$

where all the f_r's assume negative values for stress states below the yield limit, and at least one f_r is zero with no f_r positive for stress states at the yield limit. Except perhaps at the boundaries of the S_r's there is always a well-defined outwards-drawn unit normal $\underline{n}(n_i)$ to S where

$$n_i \propto \frac{\partial f}{\partial \sigma_i} \quad (i = 1,2,\ldots,I); \tag{3.4}$$

and, at a singular point of S, $\underline{n}$ is defined by

$$n_i \propto \sum_r a_r \frac{\partial f_r}{\partial \sigma_i} \quad (i = 1,2,\ldots,I), \tag{3.5}$$

where the a_r's are non-negative and not all zero, with the summation extending over all values of r for which S_r passes through the singular point in question. Relations (3.4) and (3.5) coincide at regular points, and the latter are taken as an extended definition of $\underline{n}$ that applies at all points.

The general problem of the determination of the actual form of the yield function is not discussed here, but in many important cases of practical interest the procedure is relatively straightforward. A fundamental requirement is that the yield surface is everywhere concave towards the origin. However once f is known the formulation of the flow rule on the basis of the theory of the plastic potential due to von Mises is immediate.

This theory demands that, for plastic stress states, the plastic strain-rate vector and the outwards-drawn normal vector to the yield surface at this stress state have the same direction. Otherwise the plastic strain-rate vector is identically zero. Thus

$$\left.\begin{aligned} \underline{e} &= 0 \quad \text{wherever either } f < 0 \text{ or } f = 0 \text{ and } f' < 0, \\ \underline{e} &= \lambda \underline{n} \quad \text{wherever } f = f' = 0 \ (\lambda \geq 0), \end{aligned}\right\} \quad (3.6)$$

λ being a function of position indeterminate to the extent of a constant multiplicative factor. It is unnecessary to give explicit attention to the conditions on f' (the prime denoting time differentiation) because these are automatically satisfied as the incipient stress field at collapse is supposed independent of time at least to a first approximation. Equivalently, the relations (3.6) are expressed through the relations (3.4) and (3.5) in the forms

$$\left.\begin{aligned} e_i &= 0 \text{ wherever either } f < 0 \text{ or } f = 0 \text{ and } f' < 0, \\ e_i &= \lambda \frac{\partial f}{\partial \sigma_i} \text{ wherever } f = f' = 0 \ (\lambda \geq 0), \end{aligned}\right\} (i=1,2,\ldots,I), \quad (3.7)$$

for a regular yield surface, and

$$\begin{aligned} e_i &= 0 \text{ wherever either } f_r < 0 \text{ or } f_r = 0 \text{ and } f_r' < 0 \ (r=1,2,\ldots,R), \\ e_i &= \lambda \sum_r a_r \frac{\partial f_r}{\partial \sigma_i} \text{ wherever } f_r = f_r' = 0 \text{ for at least one } r \\ &\quad (\lambda \geq 0;\ a_r = 0 \text{ if either } f_r < 0 \text{ or } f_r = 0 \text{ and } f_r' < 0, \text{ and} \\ &\quad a_r \geq 0 \text{ if } f_r = f_r' = 0), \end{aligned}$$

$$(i=1,2,\ldots,I), \quad (3.8)$$

quite generally for a singular yield surface. In relations (3.8) the a_r's are otherwise completely arbitrary, and variations in their ratios correspond to different combinations of plastic flow mechanisms.

Plastic-rigid material is incompressible, and hence the density is constant. It follows that the physical law of mass conservation is expressed in terms of a continuity condition involving only velocity components. The present theory is not applicable to problems of fracture. For our purposes fracture may be defined as severe local plastic deformation resulting in the formation of new surfaces to the body. Care is needed to distinguish between deformations which are properly regarded as being within the scope of plasticity theory and deformations involving fracture which are strictly outside the scope of this theory. Even so plasticity theory is commonly applied to the analysis of problems which involve either highly-localized regions of fracture or incipient fracture, and this practice extends considerably the application of the theory.

Finally it is necessary to introduce certain definitions in respect of discontinuities of field quantities. A surface T of isolated discontinuity is said to be of order $n(\geq 1)$ with respect to the field quantity $\mathfrak{F}$, if $\mathfrak{F}$ together with all its space derivatives up to and including those of order n-1, but not all derivatives of order n, are continuous across T. If the discontinuity effects $\mathfrak{F}$ itself then it is said to be of order zero. The basic field quantities are of course v_j, σ_i and λ.

On each side of T the various field quantities are related by equations, either differential or finite in nature, valid at all points. For example, there are the differential equations of equilibrium and the yield condition all involving the generalized stresses. In what follows it is to be understood that all equations are fundamental, and not derived, in the sense that they express the various physical laws in the simplest possible manner. Then with respect to T and each $\mathfrak{F}$, a weak discontinuity has an order not lower than the order of the highest derivative of $\mathfrak{F}$ occurring in the physical equations. Otherwise the discontinuity is called strong, and is a contact discontinuity. If the material in the immediate neighborhood remains rigid then there is said to be localized plastic flow. The above definitions need to be extended when the time enters explicitly into the problem. Further any (virtual) discontinuity is called natural or artificial according as the physical equations do or do not remain valid in its immediate neighborhood. Note that if there is a discontinuity in $\mathfrak{F}$ across T then it is properly to be regarded as the limit of a continuous distribution of $\mathfrak{F}$ which changes by a fixed amount across a narrow region enclosing T as this region everywhere shrinks up to T.

4. Analysis of the Flanges.

The flanges are treated according to a simple engineering theory of perfectly-plastic rigid beams, i.e. the flanges are approximated by one-dimensional continua. The relation between the present approximate theory and the exact theory is

reasonably conjectured to be much the same in plasticity as it is in elasticity. Exactly as in elastic beam theory, transverse shearing deformations being neglected, the fundamental kinematical assumption is that a cross-section of a flange initially normal to the neutral axis remains normal to the deformed neutral axis. If u_F and v_F denote the axial and transverse velocity components of particles lying on the neutral axis, then the rate of extension ε and of curvature $\varkappa$ of the neutral axis are given by

$$\varepsilon = \frac{du_F}{dx}, \quad \varkappa = \frac{d^2v_F}{dx^2}. \tag{4.1}$$

Let F, Q and M denote axial force, vertical shear force and bending moment. Then the rate of plastic work per unit length of flange is

$$W = F\varepsilon + M\varkappa. \tag{4.2}$$

Hence F, M, ε, $\varkappa$, u_F and $\frac{dv_F}{dx}$ are corresponding generalized stress, strain-rates and velocities, and Q is a reaction.

It is simple to establish certain general conditions of continuity on the fundamental field quantities F, Q, M, u_F and v_F. The equilibrium of an elementary slice of a flange enclosing an arbitrary fixed cross-section shows, when the thickness of this slice is made to tend to zero, that F, Q, and M are continuous. The law of mass conservation applied to the same slice, remembering that the density is con$ant and that

finite changes in the flange cross-sectional area would involve fracture, establishes that u_F is continuous. A discontinuity in v_F when properly regarded would correspond to large transverse shear strain-rates within the elementary slice. Inasmuch as the effect of transverse shear strain-rate is completely ignored in the present theory, it is accordingly necessary for consistency to assume that v_F is continuous. Thus at all cross-sections,

$$[F, Q, M, u_F, v_F] = 0, \tag{4.3}$$

the square brackets denoting discontinuity of the enclosed quantity. The sign conventions for the quantities introduced above are shown in Fig. 2.

The forces acting on an elementary slice (of length δx) of the upper flange are shown in Fig. 2. It follows that the differential equations of equilibrium are

$$\frac{dF}{dx} - t(\tau_{xy})_{y=\frac{1}{2}d} = 0, \tag{4.4}$$

$$\frac{dQ}{dx} + t(\sigma_y)_{y=\frac{1}{2}d} = 0, \tag{4.5}$$

$$\frac{dM}{dx} - Q = 0, \tag{4.6}$$

the upper flange lying along the line $y = \frac{1}{2}d$. Similar equations apply to the lower flange. Here, for simplicity and with little resulting loss in accuracy, the offset of the web edge from the flange neutral axis is neglected; this involves the omission of a term $- at(\tau_{xy})_{y=\frac{1}{2}d}$ from the left-hand-side of Eq.(4.6). If there is a distributed downwards-acting load p per unit length

of flange, then a term +p must be added to the left-hand-side of Eq.(4.5).

As established by Onat and Prager [22], the critical combination of F and M admits full plasticity at a cross-section corresponds to stress distributions over a cross-section in which axial fibers are at the yield stress either in tension ($+\sigma_o$) or in compression ($-\sigma_o$). It is straightforward to determine the form of the yield function(s) for any shape of cross-section. For example, in the case of a rectangular shape to which attention is confined here, the yield limit is expressed in terms of two yield functions, viz.

$$\left.\begin{aligned} f_1 &= F^2/F_o^2 + M/M_o - 1, \\ f_2 &= F^2/F_o^2 - M/M_o - 1, \end{aligned}\right\} \quad (4.7)$$

where

$$F_o = 2ab\sigma_o \ , \ M_o = a^2 b\sigma_o \ .$$

Here F_o and M_o are the limit axial force and moment in pure extension and bending, respectively. As will be seen in a moment the converse statement is not necessarily true. In proceeding it is necessary to modify the generalized stresses to F/F_o and M/M_o, and the generalized strain-rates to $F_o\varepsilon$ and $M_o\varkappa$. The corresponding generalized velocities can be taken as $F_o u_F$ and $M_o \frac{dv_F}{dx}$ (assuming that F_o and M_o are constant) so that $F_o\varepsilon = \frac{d}{dx}(F_o u_F)$ and $M_o\varkappa = \frac{d}{dx}(M_o \frac{dv_F}{dx})$, but these quantities are not explicitly required in this form. Then in the corresponding stress plane the yield locus is represented by two parabolic

arcs (ABC and CDA in Fig. 3) with singular points only at their common points (A and C). The flow rule is given by the following relations.

Regime	F_o	M_o	Conditions
A	2λ	$-(1-2q)\lambda$	$F/F_o=1,\ M=0;\ 0 \le q \le 1$
ABC	$2\lambda F/F_o$	λ	$-1 < F/F_o < 1,\ M > 0$
C	-2λ	$(1-2q)\lambda$	$F/F_o=-1,\ M=0;\ 0 \le q \le 1$
CDA	$2\lambda F/F_o$	$-\lambda$	$-1 < F/F_o < 1,\ M < 0$

(4.8)

The quantity $\lambda(x)$ is non-negative, and variations in $q(x)$ correspond to different combinations of the admissible plastic flow mechanisms at a singular point.

The law of mass conservation does not yield any further information that is directly relevant to the solution of the problem. It determines only the rate of change A' of cross-sectional area,

$$A' = -A\varepsilon. \tag{4.9}$$

The solution of the problem, so far as the flanges are concerned, must satisfy the continuity solutions (4.3), the equilibrium equations (4.4)-(4.6) and the flow rule (4.8). Note that the matching of the solution obtaining in either rigid or plastic regimes, i.e. at rigid-plastic or plastic-plastic interfaces is achieved through the continuity relations. The solution, if it is to be strictly within the present framework, will

not admit discontinuities in F, Q, M, u_F and v_F.

Finally it remains to discuss the general question of discontinuities in the field quantities. The highest derivatives of F, Q, M, u_F and $\frac{dv_F}{dx}$ that occur in the physical equations are of the first order. A weak discontinuity in these quantities is now considered. First a discontinuity in $\frac{dF}{dx}$ and $\frac{dQ}{dx}$ involves discontinuities in $(\tau_{xy})_{y=\frac{1}{2}d}$ and $(\sigma_y)_{y=\frac{1}{2}d}$, respectively, and must therefore be discussed later when the analysis of the web has been completed (see p. 46). A discontinuity in $\frac{dM}{dx}$ involves a discontinuity in Q which violates one of conditions (4.3), and is not admissible. Secondly discontinuities in ε and $\varkappa$ can occur only either within or at the boundary of a plastic regime. Then, remembering that F is continuous, the following results are found from the flow rule.

Regime	$[F_o \varepsilon]$	$[M_o \varkappa]$
A	$2[\lambda]$	$-[(1-2q)\lambda]$
ABC	$2(F/F_o)[\lambda]$	$[\lambda]$
C	$-2[\lambda]$	$[(1-2q)\lambda]$
CDA	$2(F/F_o)[\lambda]$	$-[\lambda]$

(4.10)

Here $[\varepsilon]$ and $[\varkappa]$ are supposed not both zero, and the results for the three distinct cases follow immediately and are tabulated below.

$[\varepsilon]$	$[\varkappa]$	Regime	Conditions
$\neq 0$	$\neq 0$	ABC, CDA	$F, [\lambda] \neq 0$
		A, C	$F, [\lambda] \neq 0$, $[(1-2q)\lambda] \neq 0$
$\neq 0$	0	A, C	$[\lambda] \neq 0$, $[(1-2q)\lambda] = 0$
0	$\neq 0$	B, D	$[\lambda] \neq 0$
		A, C	$[\lambda] = 0$, $[1-2q] \neq 0$

(4.11)

Thus if either regime ABC or CDA applies then discontinuities in axial strain-rate and curvature-rate must occur together and are not independent, save when there is zero axial force in which case no discontinuity is permissible in axial strain-rate although a discontinuity is permissible in curvature-rate; and if either regime A or C applies then discontinuities in axial strain-rate and curvature-rate may occur separately or together. At first sight the behavior in regime A or C may appear rather surprising, but it is simply explained by the fact that if all axial fibers at a section are stressed to the tensile or compressive yield stress it does not follow that there is constant axial velocity across the section - the distribution of axial velocity is only required to be a continuous linear function, either uniformly positive or negative, of the distance from the neutral axis, and hence bending as well as extension may be accommodated. This completes the discussion of weak discontinuities.

Next consider strong discontinuities. A discontinuity in F,Q or M violates equilibrium, and is therefore artificial.

Such discontinuities are not considered. Discontinuities in u_F and $\frac{dv_F}{dx}$ cannot occur within a rigid regime but may occur elsewhere. Here $[u_F]$ and $[\frac{dv_F}{dx}]$ are not both zero, and hence one of ε and κ becomes very large within the elementary slice. Now F remains continuous, and hence λ becomes very large within the elementary slice. Let

$$I = \lim \int \lambda(x)dx \ (>0), \quad J = \lim \int \left\{1-2q(x)\right\} \lambda(x)dx \ (-I \leq J \leq + I), \tag{4.12}$$

the limits being taken as the thickness of the elementary slice is made to tend to zero. The following relations now follow through integration of the flow rule.

Regime	$[F_o u_F]$	$[M_o \frac{dv_F}{dx}]$
A	$2I$	$-J$
ABC	$2(F/F_o)I$	I
C	$-2I$	J
CDA	$2(F/F_o)I$	$-I$

(4.13)

The results for the three distinct cases now follow immediately.

$[F_o u_F]$	$[M_o \frac{dv_F}{dx}]$	Regime	Conditions
$\neq 0$	$\neq 0$	ABC, CDA	$F \neq 0$
		A, C	$J \neq 0$
$\neq 0$	0	A, C	$J = 0$
0	$\neq 0$	B, D	---

(4.14)

Thus if either regime ABC or CDA applies then discontinuities in axial velocity and angular velocity must occur together and are not independent, save when there is zero axial force in which case no discontinuity is permissible in axial velocity although a discontinuity is permissible in angular velocity, and if either regime A or C applies then a discontinuity in axial velocity always occurs and may also be accompanied by an independent discontinuity in angular velocity. These discontinuities are classified in the following way. First if $[F_o u_F] = 0$ and $[M_o \frac{dv_F}{dx}] \neq 0$ then the discontinuity is immediately recognized as corresponding to a 'yield hinge' long familiar in the plastic analysis of beams. Second if $[F_o u_F] \neq 0$ and $[M_o \frac{dv_F}{dx}] \neq 0$ then the discontinuity is called an 'extensible yield hinge', and was first introduced by Onat and Prager [22] . Last if $[F_o u_F] \neq 0$ and $[M_o \frac{dv_F}{dx}] = 0$ then the discontinuity is simply called an 'extension'. A discontinuity in axial velocity involves very high rates-of-change of cross-sectional area for

$$\frac{1}{A} \lim \int A' \, dx = - [u_F] , \tag{4.15}$$

the limit being taken in the usual way. This situation corresponds to incipient fracture simultaneously with plastic flow, and is accordingly not strictly admissible within the framework of the present theory. Moreover, if discontinuities occur in either axial velocity or angular velocity then high rates of strain of axial fibers must occur. The present theory does not take account of such effects. Accordingly it must be expected that viscous effects will immediately serve to modify such sharp discontinuities. Moreover effects due to inertia and strain-

hardening will soon become important at such a section. In a general way the discontinuities will be spread out rapidly over a small length of the flange. The discontinuities are artificial, but nevertheless analysis within the present framework may reasonably be expected to furnish a sufficiently adequate description of the actual mechanical behavior except in the immediate neighborhood of such discontinuities.

5. Analysis of the Web.

The web is treated according to an approximate theory of plane stress for a hypothetical material which is unable to support compressive stresses. Thus the web is approximated by a two-dimensional continuum. So far as the assumptions of plane stress are concerned, the relation between the present approximate theory and the exact theory is reasonably conjectured to be much the same in plasticity as it is in elasticity. The generalized stresses are the mean (i.e., thickness averaged) stresses σ_x, σ_y, τ_{xy} , and the corresponding generalized strain-rates are the mean strain-rates ε_x, ε_y, γ_{xy} . If v_x and v_y denote the components of mean velocity then

$$\varepsilon_x = \frac{\partial v_x}{\partial x} , \quad \varepsilon_y = \frac{\partial v_y}{\partial y} , \quad \gamma_{xy} = \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} . \qquad (5.1)$$

The forces acting on a small element $\delta x \times \delta y \times t$ are shown in Fig. 2. The differential equations of equilibrium are therefore

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 , \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 . \qquad (5.2)$$

The rate of plastic work per unit area of middle surface is

$$W = t(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy}\gamma_{xy}). \tag{5.3}$$

The principal stresses are denoted by σ_1 and σ_2, and are distinguished through the choice

$$\sigma_1 \geq \sigma_2. \tag{5.4}$$

In the case of equality in condition (5.4) there is stress isotropy. If φ is the anti-clockwise rotation of the σ_1 - and σ_2 - directions from the x - and y - axes, respectively, then

$$\left.\begin{aligned} 2\sigma_1 &= \sigma_x + \sigma_y + r, \quad 2\sigma_2 = \sigma_x + \sigma_y - r, \\ \cos 2\varphi &= (\sigma_x - \sigma_y)/r, \quad \sin 2\varphi = 2\tau_{xy}/r, \\ r &= +\left\{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}\right\}^{\frac{1}{2}}, \end{aligned}\right\} \tag{5.5}$$

(φ is of course only determinate to within a multiple of π), and conversely

$$\left.\begin{aligned} \sigma_x &= \cos^2\varphi\,\sigma_1 + \sin^2\varphi\,\sigma_2, \\ \sigma_y &= \sin^2\varphi\,\sigma_1 + \cos^2\varphi\,\sigma_2, \\ \tau_{xy} &= \tfrac{1}{2}\sin 2\varphi\,(\sigma_1 - \sigma_2). \end{aligned}\right\} \tag{5.6}$$

The yield condition is required to express the fact that the material will not sustain pure compresion and may yield

when either principal stress achieves the yield stress σ_o in simple tension. Hence the yield condition restricts σ_1 and σ_2 to lie either between or at the values 0 and σ_o , i.e., within the square ABCD (see Fig. 4) drawn in a stress plane in which σ_1 and σ_2 are taken as rectangular Cartesian co-ordinates. However, remembering the condition (5.4), only the triangle ABC is strictly relevant.

The yield curve ABCD is imagined folded about the diagonal AC, and consists of the lines AB and BC each described twice. The diagonal AC, its end points being excluded, is not part of the yield curve. Accordingly there are two regular plastic regimes AB and BC, and three singular plastic regimes A,B and C. Thus the yield condition is expressed in terms of two yield functions, viz.

$$f_1 = -\sigma_2, \; f_2 = \sigma_1 - \sigma_o \, , \tag{5.7}$$

subject of course to condition (5.4). This yield condition is also simply expressed in terms of the stresses σ_x, σ_y and τ_{xy} as follows. Now for the regimes A, AB and B, $0 \leq \sigma_1 \leq \sigma_o$ and $\sigma_2 = 0$; and for the regimes B, BC and C, $\sigma_1 = \sigma_o$ and $0 \leq \sigma_2 \leq \sigma_o$. Then from Eqs.(5.5) and (5.7),

$$r = \begin{cases} \sigma_x + \sigma_y & \text{for regimes A, AB, B,} \\ (\sigma_o - \sigma_x) + (\sigma_o - \sigma_y) & \text{for regimes B, BC, C,} \end{cases} \tag{5.8}$$

and hence

$$\left.\begin{aligned} &(j\sigma_o - \sigma_x)(j\sigma_o - \sigma_y) = \tau^2_{xy} , \\ &\text{where} \\ &j = \begin{cases} 0 \text{ for regimes A, AB, B,} \\ 1 \text{ for regimes B, BC, C.} \end{cases} \end{aligned}\right\} \qquad (5.9)$$

Note that regime B is characterized by either value of j.

If f denote the yield condition then the flow rule is expressed by

$$\frac{\partial v_x}{\partial x} \Big/ \frac{\partial f}{\partial \sigma_x} = \frac{\partial v_y}{\partial y} \Big/ \frac{\partial f}{\partial \sigma_y} = \frac{1}{2}\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) \Big/ \frac{\partial f}{\partial \tau_{xy}} = \lambda(x,y) \geq 0, \quad (5.10)$$

the first order partial differential coefficients of f being given the usual freedom of interpretation at singular points.

The law of mass conservation determines the rate-of-change of the web thickness, or, equivalently, the mean transverse strain-rate ε_z, in terms of ε_x and ε_y:

$$t' = t\varepsilon_z = - t(\varepsilon_x + \varepsilon_y). \qquad (5.11)$$

The set of five equations (5.2), (5.9) and (5.10) in the five unknowns σ_x , σ_y , τ_{xy} , v_x and v_y form the basis for the determination of the stress and velocity fields in the plastic region. Note that the first three of these equations involve only the stresses. The other equations involve the stresses and the velocities, and are homogeneous in the latter.

In the rigid (elastic) region the equations of equilibrium remain the same, but now the strain-rates all vanish identically and the yield condition is replaced by the requirement

of compatibility,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0. \tag{5.12}$$

The set of three equations (5.2) and (5.12) in the three unknowns σ_x, σ_y and τ_{xy} form the basis for the determination of the stress field in the rigid (elastic) region.

It is necessary to establish certain conditions on the field quantities at the plastic-elastic interface. Let the interface be a simple open curve Γ with everywhere a continuously-turning tangent except perhaps at a finite number of isolated points. Let the normal and tangential directions to Γ be denoted by increasing values of local rectangular Cartesian co-ordinates n and s, respectively. The equilibrium of a small rectangular-shaped volume $\delta s \times \delta n \times t$ shows, as the thickness δn is made to tend to zero, that the normal stress σ_n and the shear stress τ_{ns} are continuous, but that the tangential stress σ_t need not be continuous (see Fig. 5). The law of mass conservation applied to the same volume, remembering that the density is constant and that finite changes in the web thickness would involve fracture which is here excluded, establishes that v_n is continuous but that v_s may be discontinuous. Thus

$$[\sigma_n, \tau_{ns};\ v_n] = 0 \tag{5.13}$$

across a plastic-elastic interface, but σ_s and v_s may be discontinuous across such an interface.

It will be established later that the directions of principal stress at any point of a plastic stress field are mathematically-preferred directions. This fact at once focuses attention on the field of principal stress trajectories as the fundamental unknown element to be determined in the solution of the problem. Let the two orthogonal families of lines of principal stress be identified through the orthogonal curvilinear co-ordinates α and β. Hence $\alpha(x,y)$ and $\beta(x,y)$ are supposed continuously differentiable functions such that

$$\frac{\partial\alpha}{\partial x}\frac{\partial\beta}{\partial x} + \frac{\partial\alpha}{\partial y}\frac{\partial\beta}{\partial y} = 0. \tag{5.14}$$

A member of one family of principal stress trajectories is then defined by an equation β = const. and is called an α-line; and similarly a member of the other family is defined by an equation α = const. and is called a β-line. The two families are distinguished through the choice

$$\sigma_\alpha = \sigma_1 , \sigma_\beta = \sigma_2 . \tag{5.15}$$

Note that the lines of principal stress are not uniquely determined in the case of stress isotropy, e.g. when regime A or C applies throughout a region.

The fundamental equations will now be developed in terms of the geometry of the α,β - co-ordinate system (see Fig. 6). Let $\psi(\alpha,\beta)$ be the angle made by the tangent to the α-line through the point $P(\alpha,\beta)$ with a fixed direction; and let $h_\alpha(\alpha,\beta)\delta\alpha$ and $h_\beta(\alpha,\beta)\delta\beta$ be the elements of length on the

α- and β-lines, respectively. Let $\varkappa_\alpha$ and $\varkappa_\beta$ be the (algebraic) curvatures of the α- and β-lines, and be defined by

$$\varkappa_\alpha = \frac{1}{\rho_\alpha} = -\frac{1}{h_\alpha}\frac{\partial\psi}{\partial\alpha}, \quad \varkappa_\beta = \frac{1}{\rho_\beta} = \frac{1}{h_\beta}\frac{\partial\psi}{\partial\beta}. \tag{5.16}$$

Simple geometry applied to the quasi-rectangular element PQRS of Fig. 7 shows that

$$\frac{\partial h_\alpha}{\partial\beta} = h_\alpha h_\beta \varkappa_\alpha, \quad \frac{\partial h_\beta}{\partial\alpha} = h_\alpha h_\beta \varkappa_\beta. \tag{5.17}$$

Further since $\frac{\partial^2\psi}{\partial\alpha\partial\beta} = \frac{\partial^2\psi}{\partial\beta\partial\alpha}$ it follows that the curvatures are related through the condition

$$\frac{\partial\varkappa_\alpha}{h_\beta\partial\beta} + \frac{\partial\varkappa_\beta}{h_\alpha\partial\alpha} + \varkappa_\alpha^2 + \varkappa_\beta^2 = 0. \tag{5.18}$$

Let v_α and v_β be the components of velocity in the directions of increasing α and β, respectively. Then it is easy to show that the corresponding strain-rates ε_α, ε_β, $\gamma_{\alpha\beta}$ are given by

$$\begin{aligned} \varepsilon_\alpha &= \frac{\partial v_\alpha}{h_\alpha\partial\alpha} + \varkappa_\alpha v_\beta, \quad \varepsilon_\beta = \frac{\partial v_\beta}{h_\beta\partial\beta} + \varkappa_\beta v_\alpha, \\ \gamma_{\alpha\beta} &= \frac{\partial v_\beta}{h_\alpha\partial\alpha} + \frac{\partial v_\alpha}{h_\beta\partial\beta} - \varkappa_\alpha v_\alpha - \varkappa_\beta v_\beta . \end{aligned} \tag{5.19}$$

The forces acting on a small element $h_\alpha\delta\alpha \times h_\beta\delta\beta \times t$ are shown in Fig. 7. The differential equations of equilibrium are accordingly

$$\frac{\partial}{\partial\alpha}(h_\beta\sigma_\alpha) - h_\alpha h_\beta\varkappa_\beta\sigma_\beta = 0, \quad \frac{\partial}{\partial\beta}(h_\alpha\sigma_\beta) - h_\alpha h_\beta\varkappa_\alpha\sigma_\alpha = 0, \tag{5.20}$$

and after use of Eqs. (5.17) these equations take the form

$$\frac{\partial\sigma_\alpha}{\partial\alpha} + h_\alpha\varkappa_\beta(\sigma_\alpha - \sigma_\beta) = 0 \ , \quad \frac{\partial\sigma_\beta}{\partial\beta} + h_\beta\varkappa_\alpha(\sigma_\beta - \sigma_\alpha) = 0. \qquad (5.21)$$

The rate of plastic work per unit area of middle surface is

$$W = t(\sigma_\alpha\varepsilon_\alpha + \sigma_\beta\varepsilon_\beta). \qquad (5.22)$$

The yield condition is given by replacing σ_1 and σ_2 by σ_α and σ_β respectively, i.e.

$$f_1 = -\sigma_\beta \ , \ f_2 = \sigma_\alpha - \sigma_o . \qquad (5.23)$$

In an isotropic material the principal axes of stress and plastic strain-rate must coincide, i.e.

$$\gamma_{\alpha\beta} = 0 \qquad (5.24)$$

throughout the plastic region. This condition imposes a very severe condition on the velocity field. Apart from relation (5.24) the flow rule is given by the following relations.

Regime	ε_α	ε_β	Conditions
A	$-(1-q)\lambda$	$-q\lambda$	$\sigma_\alpha = \sigma_\beta = 0 \ ; \ \frac{1}{2} \leq q \leq 1$
AB	0	$-\lambda$	$0 < \sigma_\alpha < \sigma_o \ , \ \sigma_\beta = 0$
B	$q\lambda$	$-(1-q)\lambda$	$\sigma_\alpha = \sigma_o \ , \ \sigma_\beta = 0 \ ; \ 0 \leq q \leq 1$
BC	λ	0	$\sigma_\alpha = \sigma_o \ , \ 0 < \sigma_\beta < \sigma_o$
C	$(1-q)\lambda$	$q\lambda$	$\sigma_\alpha = \sigma_\beta = \sigma_o \ ; \ 0 \leq q \leq \frac{1}{2}$

(5.25)

Here $\lambda(\alpha,\beta)$ is non-negative and $q(\alpha,\beta)$ is subject to the stated inequalities, but otherwise these quantities are unrestricted as yet.

The equation of continuity is

$$\frac{1}{h_\alpha}\frac{\partial v_\alpha}{\partial \alpha} + \frac{1}{h_\beta}\frac{\partial v_\beta}{\partial \beta} + \varkappa_\beta v_\alpha + \varkappa_\alpha v_\beta = -\varepsilon_z = -\frac{t'}{t}. \qquad (5.26)$$

The set of six equations (5.18), (5.21) and (5.23)-(5.25) in the six unknowns σ_α, σ_β, $\varkappa_\alpha$, $\varkappa_\beta$, v_α and v_β form the basis for the determination of the stress and velocity fields in the plastic region. Note that the first four of these equations, viz. Eqs.(5.18), (5.21) and (5.23) involve only the quantities σ_α, σ_β, $\varkappa_\alpha$ and $\varkappa_\beta$. The other two equations involve the velocities and are homogeneous in these quantities.

In the rigid (elastic) region the equations of equilibrium remain the same, but now the strain-rates all vanish and the yield condition is replaced by the requirement of compatibility

$$\left\{\frac{\partial}{\partial\alpha}\left(\frac{h_\beta}{h_\alpha}\frac{\partial}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{h_\alpha}{h_\beta}\frac{\partial}{\partial\beta}\right)\right\}(\sigma_\alpha + \sigma_\beta) = 0. \qquad (5.27)$$

The set of four equations (5.18), (5.21) and (5.27) form the basis for the determination of the four unknowns σ_α, σ_β, $\varkappa_\alpha$ and $\varkappa_\beta$ throughout the rigid (elastic) region.

An essential preliminary to devising methods of integration of the plastic equations is to determine their mathematical nature. The rigid (elastic) equations are of course elliptic in type. Suppose that the stress and velocity fields are known throughout the plastic region bounded by a simple closed curve Γ (see Fig. 8). For simplicity it is assumed that Γ has everywhere a continuously-turning tangent, but this

condition may easily be relaxed to the simpler assumption that Γ is composed of a finite number of arcs with the above property. Let P be a typical point of Γ; and let n and s denote local rectangular Cartesian co-ordinates with origin P, the directions of increasing n and s being along the outwards-drawn normal and positive tangent to Γ. In what follows we shall consider the set of equations found by replacing x and y by n and s, respectively, throughout Eqs.(5.2), (5.9) and (5.10). The form of the physical equations is of course unaltered by the choice of this new system of co-ordinates.

Now the values of the components of the stress and velocity fields (satisfying the plastic equations) are known throughout the interior of Γ. Hence the values of these quantities together with their partial differential coefficients of all orders - the data being supposed analytic - are all known inside Γ. We now ask if the known data just inside Γ, together with the supposition that the region just outside Γ is plastic, is sufficient to determine uniquely the corresponding data just outside Γ. If this is so then the the known solution inside Γ may be continued a small way into the region outside Γ, and so on. If this is not so, save when additional data are assigned, then Γ will be a characteristic. The characteristic condition corresponds of course to the situation when the field quantities together with their normal and tangential derivatives are not necessarily all continuous across Γ. In other words different analytic solutions of the plastic equations can only

touch along a characteristic, and the contact may be of any order.

First note that the results (5.13) apply to Γ, i.e. σ_n, τ_{ns} and v_n are continuous across Γ, but σ_s and v_s may be discontinuous across Γ. The values, say of σ_s, just inside and just outside Γ will be denoted by σ_s^- and σ_s^+, respectively; and the discontinuity measured positively outwards across Γ, i.e. $\sigma_s^+ - \sigma_s^-$, is denoted by $[\sigma_s]$; and similarly for other quantities.

a) The stress equations.

The stress equations will be considered first. The yield condition (see Eq.(5.9))

$$(j\sigma_o - \sigma_n)(j\sigma_o - \sigma_s) = \tau_{ns}^2 \tag{5.28}$$

applies on both sides of Γ, and hence as τ_{ns} is continuous

$$[(j\sigma_o - \sigma_n)(j\sigma_o - \sigma_s)] = 0. \tag{5.29}$$

The analysis now proceeds differently according as the same value of j does or does not apply on both sides of Γ.

Case (i) : $j^- = j^+ (= j)$. Equation (5.29) shows that

$$(\sigma_n - j\sigma_o)[\sigma_s] = 0. \tag{5.30}$$

Now if the regime be either A or C, so that there is a homogeneous isotropic stress field, then nothing more remains to be said. Attention may therefore be confined to regimes AB, B and BC. Then σ_s is continuous if $\sigma_n \neq j\sigma_o$ but is not proven continuous if

$$\sigma_n = j\sigma_o \tag{5.31}$$

when $\tau_{ns} = 0$ and hence σ_n and σ_s are principal stresses.

Suppose for the moment that there are strong discontinuities in the stress field so that Eq.(5.31) applies. Therefore if the regime is AB or BC then Γ must be an α- or a β-line, respectively; and if the regime is B then Γ must be either an α- or a β-line. The stresses σ_n and τ_{ns}, together with their tangential derivatives $\frac{\partial\sigma_n}{\partial s}$ and $\frac{\partial\tau_{ns}}{\partial s}$, are continuous across Γ. The equations of equilibrium (see Eqs.(5.2))

$$\frac{\partial\sigma_n}{\partial n} + \frac{\partial\tau_{ns}}{\partial s} = 0 , \quad \frac{\partial\tau_{ns}}{\partial n} + \frac{\partial\sigma_s}{\partial s} = 0 \tag{5.32}$$

apply on both sides of Γ, and prove that $\frac{\partial\sigma_n}{\partial n}$ is continuous. Further from Eqs.(5.28) and (5.31)

$$(\sigma_s - j\sigma_o)\frac{\partial\sigma_n}{\partial n} = 0. \tag{5.33}$$

In regimes AB, B and BC, the principal stresses are unequal, and hence $\sigma_s \neq j\sigma_o$. Therefore

$$\frac{\partial\sigma_n}{\partial n} = 0. \tag{5.34}$$

The quantities $\frac{\partial\sigma_s}{\partial s}$, $\frac{\partial\sigma_s}{\partial n}$ and $\frac{\partial\tau_{ns}}{\partial n}$ are not proven continuous, and from Eqs.(5.32) their discontinuities are restricted only by

$$\left[\frac{\partial\tau_{ns}}{\partial n}\right] + \left[\frac{\partial\sigma_s}{\partial s}\right] = 0. \tag{5.35}$$

Results governing higher order discontinuities are easily found.

Now suppose that weak discontinuities at most occur in the stress field. The stress σ_s must now be prescribed continuous when Eq.(5.31) applies. All three stresses σ_n, σ_s and τ_{ns}, together with their tangential derivatives $\frac{\partial\sigma_n}{\partial s}$, $\frac{\partial\sigma_s}{\partial s}$ and $\frac{\partial\tau_{ns}}{\partial s}$, are continuous across Γ. Equations (5.32) prove that the normal derivatives $\frac{\partial\sigma_n}{\partial n}$ and $\frac{\partial\tau_{ns}}{\partial n}$ are continuous across Γ. The only remaining first order derivative of the stresses not yet known is $\frac{\partial\sigma_s{}^+}{\partial n}$, and is given from Eq.(5.28) by

$$(\sigma_n - j\sigma_o)\left[\frac{\partial\sigma_s}{\partial n}\right] = 0. \qquad (5.36)$$

The conclusions previously established in respect of Γ for σ_s - discontinuities therefore also apply for $\frac{\partial\sigma_s}{\partial n}$ - discontinuities. Results governing higher order discontinuities are easily found. If now in addition $\frac{\partial\sigma_s}{\partial n}$ is prescribed continuous when Eq.(5.31) applies, then the lowest order partial differential coefficient which admits discontinuities is $\frac{\partial^2\sigma_s}{\partial n^2}$, and this only when Eq.(5.31) applies.

Thus it appears quite generally that discontinuities can occur only if the characteristic condition (5.31) is satisfied. The stress equations are parabolic for regimes AB and BC with the α- and β-lines, respectively, as characteristics; and are hyperbolic for regime B with the α- and β-lines as characteristics.

Case (ii) : $j^- \neq j^+$. Equation (5.29) shows that

$$\sigma_o(\sigma_o-\sigma_n-\sigma_s{}^-) + [j]\,(\sigma_n - j^+\sigma_o)[\sigma_s] = 0. \qquad (5.37)$$

The detailed analysis is omitted, and only the essential results will be stated. Note first that regimes A and C cannot meet because this would involve the discontinuity of both principal stresses. If Γ is not in regime B then a strong discontinuity in σ_s of known amount always occurs and is accompanied by weak discontinuities in the stresses also of known amounts. If Γ is in regime B then a strong discontinuity in σ_s and weak discontinuities in the stresses may occur.

b) The velocity equations.

The velocity equations will now be considered. Let the two orthogonal families of lines of maximum shear stress be identified through the orthogonal curvilinear co-ordinates γ and δ. A member of one family is then defined by an equation δ = const. and is called a γ-line; and similarly a member of the other family is defined by an equation γ = const. and is called a δ-line. Then at all points of the stress field, let the directions of increasing γ and δ be the internal and external bisectors of the directions of increasing α and β.

It has been proved that, although v_n is continuous across Γ, v_s is not necessarily continuous across Γ. Now such a discontinuity in v_s when properly regarded must correspond to high shear strain-rates near Γ. Hence as the principal axes of stress and strain-rate coincide, Γ is either a γ- or a δ-line.

Suppose for the moment that there are strong discontinuities in the velocity field so that Γ is either a γ- or a δ-line.

The velocity component v_n, together with its tangential derivative $\frac{\partial v_n}{\partial s}$, are continuous across Γ. The flow rule (see Eqs.(5.9) and (5.10)), except at a singular point, is

$$\frac{\partial v_n}{\partial n}/(j\sigma_o-\sigma_s) = \frac{\partial v_s}{\partial s}/(j\sigma_s-\sigma_n) = \frac{1}{2}(\frac{\partial v_s}{\partial n} + \frac{\partial v_n}{\partial s})/\tau_{ns} \qquad (5.38)$$

which implies that

$$(\frac{\partial v_s}{\partial n} + \frac{\partial v_n}{\partial s})/(\frac{\partial v_n}{\partial n} - \frac{\partial v_s}{\partial s}) = 2\tau_{ns}/(\sigma_n - \sigma_s). \qquad (5.39)$$

Equation (5.39) is of course the condition that the principal axes of strain-rate and stress coincide. Now in the present case

$$\sigma_n = \sigma_s , \qquad (5.40)$$

and hence from Eq.(5.39)

$$\frac{\partial v_n}{\partial n} = \frac{\partial v_s}{\partial s} . \qquad (5.41)$$

The continuity of σ_n correctly implies that of σ_s. The law of mass conservation (see Eq.(5.11)) gives

$$\frac{\partial v_n}{\partial n} + \frac{\partial v_s}{\partial s} + \varepsilon_z = 0, \qquad (5.42)$$

and hence from Eq.(5.41)

$$\frac{\partial v_n}{\partial n} = \frac{\partial v_s}{\partial s} = - \frac{1}{2}\varepsilon_z. \qquad (5.43)$$

Therefore

$$[\frac{\partial v_n}{\partial n}] = [\frac{\partial v_s}{\partial s}] = - \frac{1}{2}[\varepsilon_z] , \qquad (5.44)$$

and so $\frac{\partial v_n}{\partial n}$, $\frac{\partial v_s}{\partial s}$ and ε_z are all either continuous or discontinuous. Accordingly the quantities $\frac{\partial v_s}{\partial n}$, $\frac{\partial v_s}{\partial s}$ and $\frac{\partial v_n}{\partial n}$ are not proven continuous, and their discontinuities are restricted only by Eq.(5.44). The discussion of the stress equations has shown that if the same value of j applies on both sides of Γ, then there can be no associated discontinuities in the stress field. Otherwise Eq.(5.37), remembering that σ_s is continuous, shows that $\sigma_o = \sigma_n + \sigma_s = \sigma_\alpha + \sigma_\beta$ and hence Γ is in regime B. Weak discontinuities may now occur in the stress field.

Now suppose that weak discontinuities at most occur in the velocity field. The velocity v_s must now be prescribed continuous when Γ is a γ- or δ-line. The two velocity components v_n and v_s , together with their tangential derivatives $\frac{\partial v_n}{\partial s}$ and $\frac{\partial v_s}{\partial s}$, are now continuous across Γ , and it follows from Eq.(5.42) that $\frac{\partial v_n}{\partial n}$ and ε_z are either both continuous or discontinuous across Γ . More precisely,

$$\left[\frac{\partial v_n}{\partial n}\right] = - [\varepsilon_z] \ . \tag{5.45}$$

The only remaining first order partial differential coefficients of the velocity components not yet considered is $\frac{\partial v_s}{\partial n}$, and information concerning this quantity must be found from the flow rule. Now the flow rule (5.38) may be re-written in the form

$$\frac{\partial v_n}{\partial n} = \lambda(j\sigma_o-\sigma_s) \ , \ \frac{\partial v_s}{\partial s} = \lambda(j\sigma_o-\sigma_n) \ , \ \frac{1}{2}\left(\frac{\partial v_s}{\partial n} + \frac{\partial v_n}{\partial s}\right) = \lambda\tau_{ns} \tag{5.46}$$

where λ (non-zero) is an undetermined factor of proportionality. Since τ_{ns} and $\frac{\partial v_n}{\partial s}$ are continuous across Γ, the third of Eqs.(5.46) shows that $\frac{\partial v_s}{\partial n}$ and λ are both either continuous or discontinuous across Γ. More precisely,

$$[\frac{\partial v_s}{\partial n}] = 2\tau_{ns} [\lambda] . \tag{5.47}$$

Let us assume for the moment that ε_z is continuous across Γ. Then as both $\frac{\partial v_s}{\partial s}$ and $\frac{\partial v_n}{\partial n}$ are continuous, the first and second of Eqs.(5.46) show that $\lambda(j\sigma_o-\sigma_s)$ and $\lambda(j\sigma_o-\sigma_n)$ are continuous. Now σ_n is continuous and so

$$\sigma_o[\lambda j] = [\lambda\sigma_s] = \sigma_n[\lambda] . \tag{5.48}$$

The quantities j and σ_s are not proven continuous. However if j is continuous then the characteristic condition (5.31) applies, and Γ is an α- or a β-line depending on the particular plastic regime involved; and if σ_s is continuous then $\sigma_n = \sigma_s$ and Γ is either a γ- or a δ-line. On the other hand if ε_z is discontinuous across Γ, then $\frac{\partial v_n}{\partial n}$ is discontinuous and $\frac{\partial v_s}{\partial s}$ is continuous as before. In this case

$$[\lambda(j\sigma_o-\sigma_s)] = -[\varepsilon_z] , \quad [\lambda(j\sigma_o-\sigma_n)] = 0 \tag{5.49}$$

or, alternatively,

$$\sigma_o[\lambda j] = [\lambda\sigma_s - \varepsilon_z] = \sigma_n[\lambda] . \tag{5.50}$$

Again the quantities j and σ_s are not proven continuous. However if j is continuous then the characteristic condition (5.31) still

applies; and if σ_s is continuous then

$$(\sigma_s-\sigma_n)[\lambda] = [\varepsilon_z] \ , \tag{5.51}$$

and no further statement can be made. It has therefore been proved that strong discontinuities in the stress field may be associated with weak discontinuities in the velocity field.

The essential results are therefore as follows. Strong discontinuities in the velocity field can occur only across a γ- or δ-line, and are not associated with strong discontinuities in the stress field. Weak discontinuities in the velocity field may or may not be associated with discontinuities in the stress field. In particular if attention is confined to just one plastic regime then in the former case the characteristics of the stress and velocity equations coincide whereas in the latter case the velocity equations are hyperbolic with the γ- and δ-lines as characteristics.

Now that the existence of discontinuities in the velocity field has been established it is more straightforward to approach the question in terms of the geometry of the α,β - co-ordinate system. In all cases, for simplicity, attention is confined to the case when the same plastic regime applies on both sides of Γ.

The following results now follow from the flow rule (5.25), and govern weak discontinuities in the velocity field.

Regime	$[\varepsilon_\alpha]$	$[\varepsilon_\beta]$
A	$-[(1-q)\lambda]$	$-[q\lambda]$
AB	0	$-[\lambda]$
B	$[q\lambda]$	$-[(1-q)\lambda]$
BC	$[\lambda]$	0
C	$[(1-q)\lambda]$	$[q\lambda]$

(5.52)

Here it is supposed that $[\varepsilon_\alpha]$ and $[\varepsilon_\beta]$ are not both zero, and the results for the three cases follow immediately.

$[\varepsilon_\alpha]$	$[\varepsilon_\beta]$	Regime	Conditions
$\neq 0$	$\neq 0$	A,B,C	$[\lambda] \neq [q\lambda] \neq 0$
$\neq 0$	0	A,C	$[\lambda] \neq 0$, $[q\lambda] = 0$
		B	$[\lambda] = [q\lambda] \neq 0$
		BC	$[\lambda] \neq 0$
0	$\neq 0$	A,C	$[\lambda] = [q\lambda] \neq 0$
		AB	$[\lambda] \neq 0$
		B	$[\lambda] \neq 0$, $[q\lambda] = 0$

(5.53)

Thus if discontinuities in both ε_α and ε_β occur then regimes AB and BC are excluded; if a discontinuity occurs only in ε_α then regime AB is excluded; and if a discontinuity occurs only in ε_β then regime BC is excluded. The conditions under which such weak discontinuities in the velocity field may occur have been discussed previously.

Now consider strong discontinuities in the velocity field. Here it is supposed that $[v_\alpha]$ and $[v_\beta]$ are not both zero, and hence at least one of ε_α and ε_β becomes very large near Γ. Thus λ must become very large near Γ. Let

$$I = \lim \int \lambda(n)dn \ (> 0), \ J = \lim \int q(n)\ \lambda(n)dn \ (0 \leq J \leq I) \tag{5.54}$$

where the limit is taken as the thickness of a small region, enclosing Γ and across which the velocity changes sharply, is made to tend to zero. Now Γ is either a γ- or a δ-line, and accordingly the results are as follows subject to a certain convention to be described below.

Regime	$\pm[v_\alpha]/\sqrt{2}$	$\pm[v_\beta]/\sqrt{2}$
A	$-(I-J)$	$-J$
AB	0	$-I$
B	J	$-(I-J)$
BC	I	0
C	$I-J$	J

(5.55)

If the direction of increasing n lies in the first, second, third or fourth gradrants formed by the α- and β-lines then the quantities $[v_\alpha]/\sqrt{2}$, $[v_\beta]/\sqrt{2}$ are to be prefixed by both positive, negative and positive, both negative, and positive and negative signs, respectively. The results for the three cases follow immediately.

$[v_\alpha]$	$[v_\beta]$	Regime	Conditions
$\neq 0$	$\neq 0$	A,B,C	$I \neq J \neq 0$
$\neq 0$	0	A,C	$I \neq 0,\ J = 0$
		B	$I = J \neq 0$
		BC	$I \neq 0$
0	$\neq 0$	A,C	$I = J \neq 0$
		AB	$I \neq 0$
		B	$I \neq 0,\ J = 0$

(5.56)

Thus if discontinuities in both v_α and v_β occur then regimes AB and BC are excluded; if a discontinuity occurs only in v_α then regime AB is excluded; and if a discontinuity occurs only in v_β then regime BC is excluded. In all cases the integration of the equation of continuity across the line of discontinuity shows that

$$\frac{1}{\sqrt{2}} (\pm [v_\alpha] \pm [v_\beta]) = -\lim \int \varepsilon_z dn , \qquad (5.57)$$

the sign convention being as before and the limit being taken in the usual way. Thus as previously noted, ε_z is not in general finite. The remarks made following Eq.(4.15) are again applicable

This completes the discussion of discontinuities in the velocity equations.

It is now possible to take up the point in the analysis of the flange equations where discontinuities in $\frac{dF}{dx}$ and $\frac{dQ}{dx}$ were seen to lead to discontinuities in $(\tau_{xy})_{y=\frac{1}{2}d}$ and $(\sigma_y)_{y=\frac{1}{2}d}$,

respectively (see p. 22). Clearly the former discontinuity now appears to be inadmissible whereas the latter discontinuity is admissible if the web edge is an α- or a β-line depending on the plastic regime. Finally it is necessary to consider discontinuities in the flange axial and angular velocities. If u_F is discontinuous then the velocity component v_x in the web must also be discontinuous. This can only be accommodated through an infinity in the transverse strain-rate ε_z satisfying

$$[v_x] = - \lim \int \varepsilon_z dx \ , \tag{5.58}$$

the limit being taken in the usual sense. If $\frac{dv_F}{dx}$ is discontinuous then the quantity $\frac{\partial v_y}{\partial x}$ in the web must also be discontinuous. Such weak velocity discontinuities in the web have already been discussed. In particular if the plastic regime is continuous then the web edge is an α- or a β-line depending on the plastic regime.

The general problem of the integration of the web equations subject to various types of boundary conditions will not be examined here. However this Section will be concluded by showing that the stress and velocity fields possess certain very striking properties. These may be compared with certain results for plastic plane strain fields (see [11]).

Consider first the regular plastic regimes.

1) <u>Regime AB: $0 < \sigma_\alpha < \sigma_o$, $\sigma_\beta = 0$.</u>

The first of Eqs.(5.20) is immediately integrable, and shows that $h_\beta \sigma_\alpha$ is at most a function of β only; i.e. $h_\beta \sigma_\alpha$ is

constant along an α-line, but in general the value of this constant will vary from one α-line to another. The other equation shows that $\varkappa_\alpha$ is zero, i.e. the α-lines are straight. Therefore $\varkappa_\beta = \frac{1}{h_\beta}\psi'(\beta)$, and hence $\rho_\beta\sigma_\alpha$ is constant along an α-line. The geometry of the stress field is therefore particularly simple, and is shown in Fig. 9(a). Conversely any geometric field of this type may be associated with a stress field. The flow rule equations (5.24) and (5.25) show that the α-lines are lines of zero rate-of-extension and that

$$\varepsilon_\beta = \frac{\partial v_\beta}{h_\beta \partial \beta} + \varkappa_\beta v_\alpha \leq 0, \quad \gamma_{\alpha\beta} = \frac{\partial v_\alpha}{h_\beta \partial \beta} + \frac{\partial v_\beta}{h_\alpha \partial \alpha} - \varkappa_\beta v_\beta = 0. \qquad (5.59)$$

The α-lines are said to form a 'fan'. If the evolute of the α- lines degenerates to a point then the fan is described as 'centered'. In this case choose α and β to be polar co-ordinates r and θ as shown in Fig. 9(b). Thus

$$\alpha = r, \ \beta = \theta, \ h_\alpha = 1, \ h_\beta = r, \ \varkappa_\alpha = 0, \ \rho_\beta = r \ ; \qquad (5.60)$$

and the stresses and velocities are given by

$$\left.\begin{aligned} r\sigma_r &= f(\theta)\ , \ \sigma_\theta = 0\ , \\ v_r &= g(\theta)\ , \ v_\theta = g'(\theta) + rh(\theta)\ , \end{aligned}\right\} \qquad (5.61)$$

where f, g and h are arbitrary functions subject only to

$$g''(\theta) + g(\theta) + rh'(\theta) \leq 0. \qquad (5.62)$$

To sum up: the α-lines are characteristics for the stress equations, and are straight lines with zero rate-of-

extension; and along an α-line the stress σ_α is proportional to the curvature of the β-lines where they cross this α-line.

2) Regime BC: $\sigma_\alpha = \sigma_o$, $0 < \sigma_\beta < \sigma_o$.

It is straightforward to obtain results similar to those above. It is found that the β-lines are straight, and that $\rho_\alpha(\sigma_o-\sigma_\beta)$ is constant along a β-line, the value of this constant in general varying from one line to another. The geometry of the stress field is shown in Fig. 10(a). The flow rule shows that the β-lines are lines of zero rate-of-extension, and that

$$\varepsilon_\alpha = \frac{\partial v_\alpha}{h_\alpha \partial\alpha} + \varkappa_\alpha v_\beta \geq 0 \ , \ \gamma_{\alpha\beta} = \frac{\partial v_\alpha}{h_\beta \partial\beta} + \frac{\partial v_\beta}{h_\alpha \partial\alpha} - \varkappa_\alpha v_\alpha = 0 \qquad (5.63)$$

The β-lines form a fan, and if this degenerates to a centered fan (see Fig. 10(b)) then the stresses and velocities are given in terms of polar co-ordinates r and Θ by

$$\left.\begin{aligned} r(\sigma_o-\sigma_r) &= f(\Theta) \ , \ \sigma_\Theta = \sigma_o \ , \\ v_r = g(\Theta), \ v_\Theta &= g'(\Theta) + rh(\Theta), \end{aligned}\right\} \qquad (5.64)$$

where f, g and h are arbitrary functions subject only to

$$g''(\Theta) + g(\Theta) + rh'(\Theta) \geq 0 \qquad (5.65)$$

To sum up: the β-lines are characteristics for the stress equations, and are straight lines with zero rate-of-extension; and along a β-line the modified stress $\sigma_o-\sigma_\beta$ is proportional to the curvature of the α-lines where they cross this β-line.

Now consider the singular plastic regimes.

3) <u>Regime A: $\sigma_\alpha = \sigma_\beta = 0$.</u>

All points in the stress field are isotropic, and there are no unique lines of principal stress.

4) <u>Regime B: $\sigma_\alpha = \sigma_0$, $\sigma_\beta = 0$.</u>

The equilibrium equations (5.20) show that $\varkappa_\alpha$ and $\varkappa_\beta$ are both zero, i.e. the α- and β-lines are straight. The lines of principal stress therefore form a rectangular mesh.

5) <u>Regime C: $\sigma_\alpha = \sigma_\beta = \sigma_0$.</u>

All points in the stress field are isotropic, and there are no unique lines of principal stress.

6. <u>Wagner Tension-Field Beam Theory</u>.

It will now be shown that Wagner's tension-field beam analysis [2] is a special case of the analysis developed in the previous Sections. For simplicity attention will be confined to the beam ABCD of Fig. 11 whose function is to transfer shear load from the rigid section AB to the fixed rigid section CD. The flange connections at A,B,C and D are supposed to be pin-jointed. Wagner's theory determines the limit load when the design is such that collapse occurs through failure only of the web.

Let the angular velocity of both flanges be ω. Consider the following velocity field for the web:

$$v_x = 0 \ , \ v_y = -\omega x. \tag{6.1}$$

Then

$$\varepsilon_x = \varepsilon_y = 0 \, , \; \gamma_{xy} = - \omega \, , \tag{6.2}$$

and hence the principal axes of strain-rate (and stress) make angles of 45° with the horizontal and vertical directions. Thus

$$\varepsilon_\alpha = \tfrac{1}{2} \omega, \; \varepsilon_\beta = - \tfrac{1}{2} \omega, \; \gamma_{\alpha\beta} = 0, \tag{6.3}$$

and as $\omega > 0$ it follows that the web is in regime B, i.e.,

$$\sigma_\alpha = \sigma_o \, , \; \sigma_\beta = 0. \tag{6.4}$$

In other words, the waves run at 45° to the horizontal. Further from Eqs.(6.4) it follows that

$$\sigma_x = \sigma_y = - \tau_{xy} = \tfrac{1}{2} \sigma_o \, . \tag{6.5}$$

The equations for the forces and bending moments in the flanges are

$$\frac{dF}{dx} \pm \frac{1}{2} t\sigma_o = 0, \; \frac{dQ}{dx} \pm \frac{1}{2} t\sigma_o = 0, \; \frac{dM}{dx} - Q = 0, \tag{6.6}$$

the upper and lower signs corresponding to the upper and lower flanges, respectively. The horizontal pull by the web on the end AB is $\frac{1}{2} \sigma_o td$, and this must be reacted by compressive forces $\frac{1}{4}\sigma_o td$ in each flange. Remembering that $M(0) = M(\ell) = 0$ for each flange it then follows from Eqs.(6.6) that

$$\left.\begin{aligned} F &= \pm \tfrac{1}{2}\, t\sigma_o(\ell - x) - \tfrac{1}{4}\,\sigma_o td, \\ Q &= \pm \tfrac{1}{4}\, t\sigma_o(\ell - 2x), \\ M &= \pm \tfrac{1}{4}\, t\sigma_o\, x(\ell - x). \end{aligned}\right\} \tag{6.7}$$

The downwards-acting shear force applied to the end section AB is

$$P = \tfrac{1}{2}\,\sigma_o td. \tag{6.8}$$

The velocity field (6.2) requires that the flanges are rigid, and hence the yield condition for the flanges is not violated. Now M is non-negative and non-positive for the upper and lower flanges, respectively. Thus it is necessary that

$$F^2/F_o^{\,2} \pm M/M_o - 1 \leq 0, \tag{6.9}$$

i.e.

$$t^2\left\{\pm(\ell - x) - \tfrac{1}{2}d\right\}^2/16a^2b^2 + tx(\ell - x)/4a^2b - 1 \leq 0. \tag{6.10}$$

The maximum value of the left-hand-side of (6.10) occurs when

$$x/\ell = \left\{1 - \frac{t}{2b}\left(1 \mp \frac{1}{2}\frac{d}{\ell}\right)\right\}/(2 - t/2b). \tag{6.11}$$

Thus if

$$\left\{1 - \frac{t}{4b}\left(1 - \frac{\ell^2}{4a^2}\right) + \frac{t^2}{16a^2b^2}\left(\frac{1}{4}d^2 \mp \frac{1}{2}d\ell\right)\right\} / \left(1 - \frac{t}{4b}\right) > 0 \tag{6.12}$$

then the beam will fail in shear at the load given by Eq.(6.8).

An important extension would be to apply the present analysis to tapered tension-field beams.

7. Limits of Economy of Material in a Tension-Field Beam.

Under a given type of loading a given structure usually fails in a well-defined way. A minimum weight structure, however, is capable of failing in any one of a number of ways. The limits of economy of material in structures have not yet been investigated in a systematic and comprehensive manner. Some references to previous work have been given by Hopkins and Prager [23].

One of the immediate steps required in the development of the present investigation is to discuss the limits of economy of material in a tension-field beam. Some brief introductory remarks to this problem will now be given.

The present discussion will be limited to the beam shown in Fig. 11. This beam involves three structural elements, viz. the upper and lower flanges and the web. The type of failure that occurs in the beam depends upon the relative strength of these elements. For example, if the web is relatively weak in comparison with the flanges, then the beam will fail in shear according to the Wagner theory as described in the previous Section.

Let

γ = weight of material per unit volume;

G = total weight of beam; and

A_c , A_t = compression and tension flange cross-sectional areas.

Then

$$G = \gamma \ell dt \left\{ (A_c + A_t)/dt + 1 \right\}, \qquad (7.1)$$

it being assumed that A_c and A_t are constant. In the case discussed in Section 6 the limit shear load P is $\frac{1}{2} \sigma_o dt$, and hence Eq.(7.1) can be written in the form

$$G = \gamma \ell^3 \left(\frac{2P}{\sigma_o \ell^2}\right) \left\{ (A_c + A_t)/dt + 1 \right\}. \qquad (7.2)$$

More generally the weight may be written in the form

$$G = \alpha \gamma \ell^3 \left(\frac{2P}{\sigma_o \ell^2}\right), \qquad (7.3)$$

and the coefficient α will be called the weight factor. The values of γ, P and ℓ being fixed, the present problem is to determine the values of A_c , A_t , d and t which render the weight factor a minimum. The relations (6.12) impose two conditions. Other conditions must be found through the study of competive modes of collapse, e.g. those in which either the upper or the lower flange remains rigid at collapse of the beam. The general procedure may be expected to parallel quite closely the analysis given in Ref. [23], and no further discussion of the problem is attempted here.

Acknowledgement

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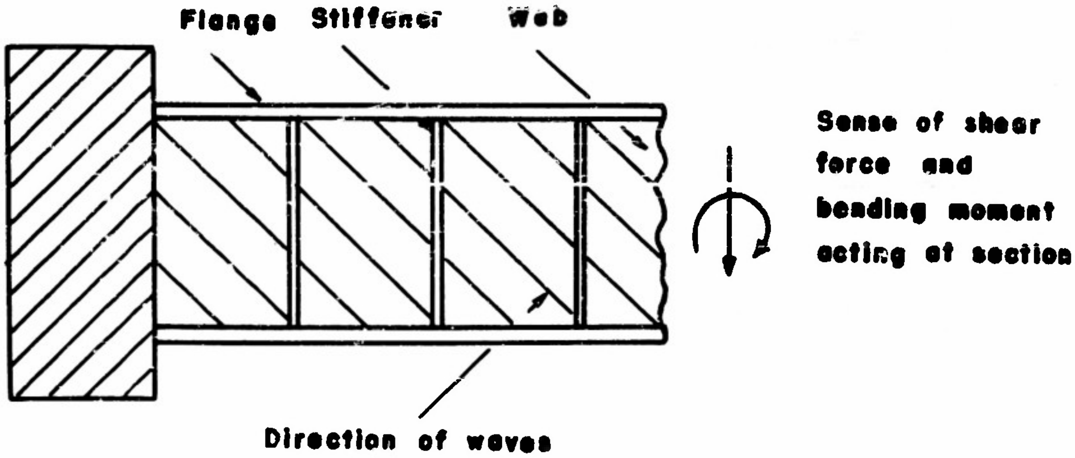


Fig. 1 Tension-field beam

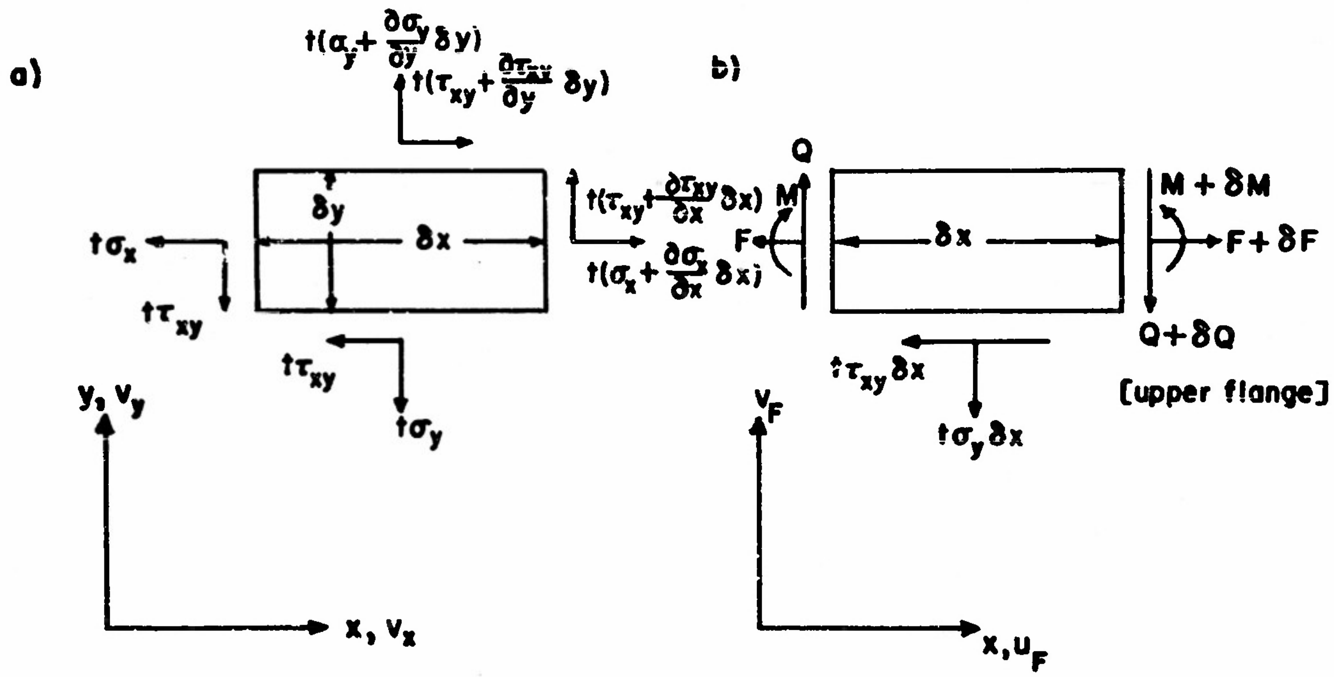


Fig. 2 Forces acting on small element of a) the web and b) the flanges

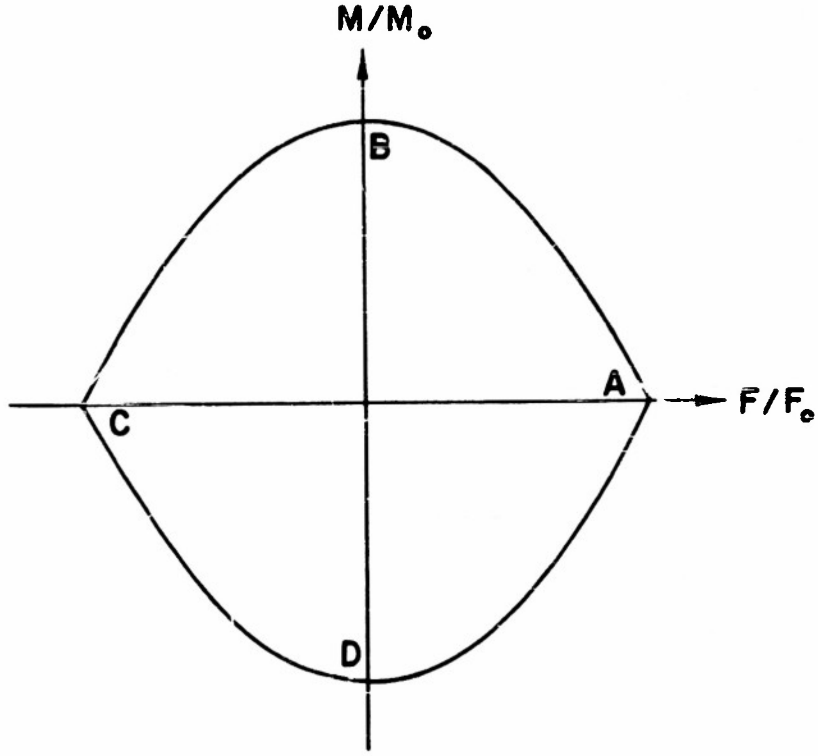


Fig. 3 Yield locus for flanges

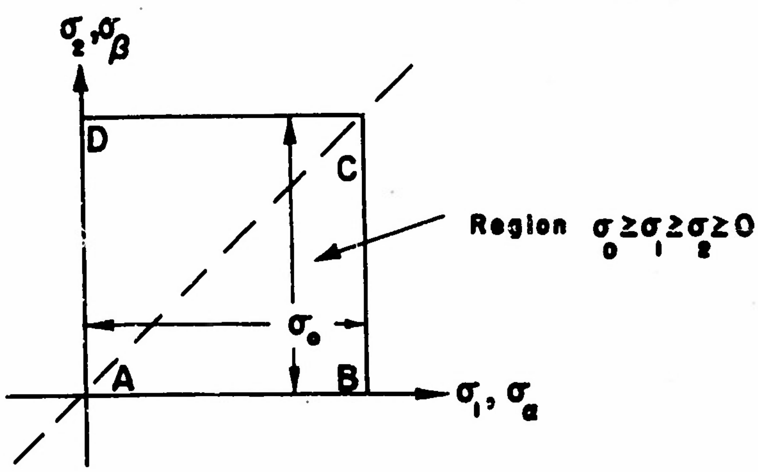


Fig. 4 Yield locus for web

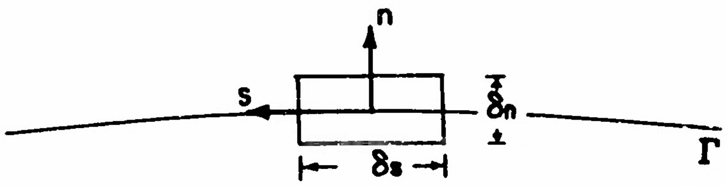


Fig. 5 Co-ordinates on discontinuity surface

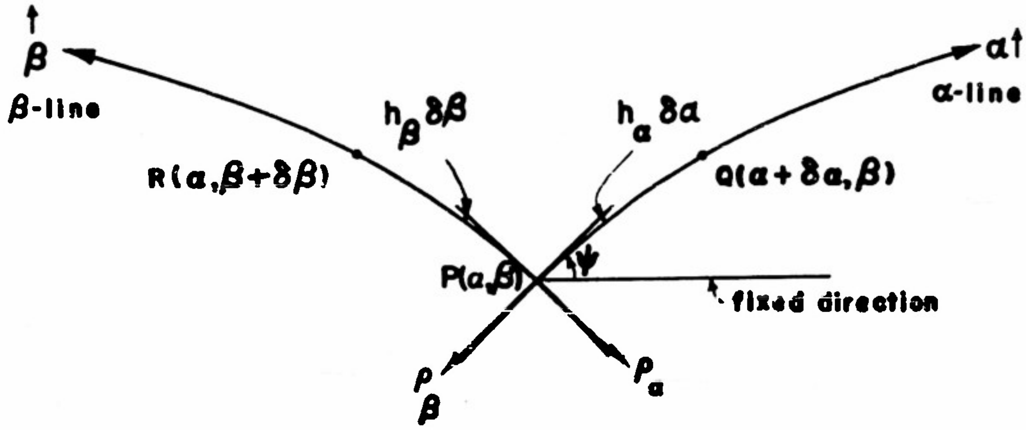


Fig. 6 Geometry of the α,β–co-ordinate system

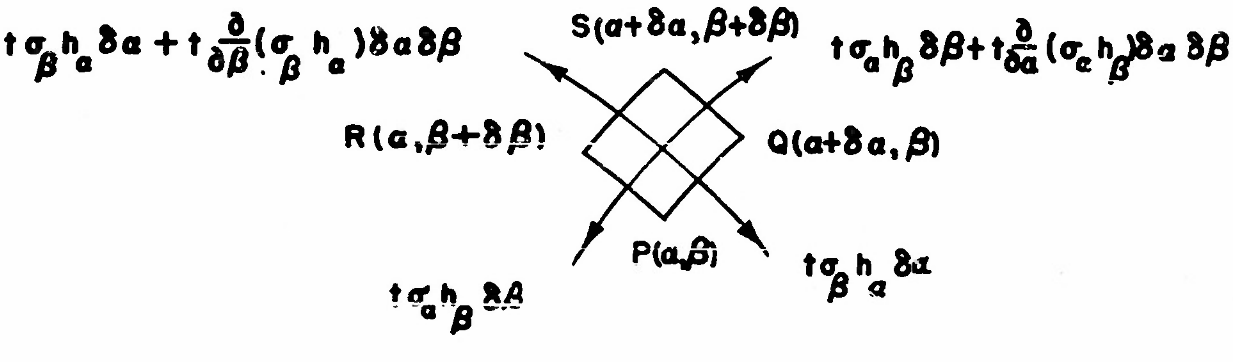


Fig. 7 Forces acting on small element of web

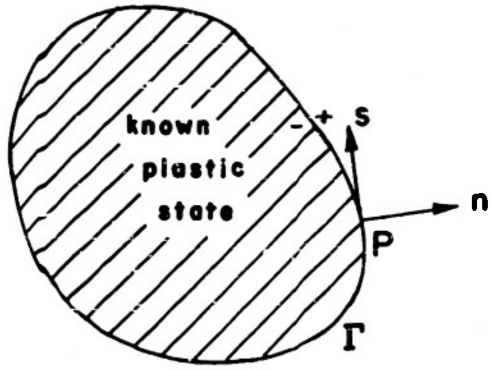
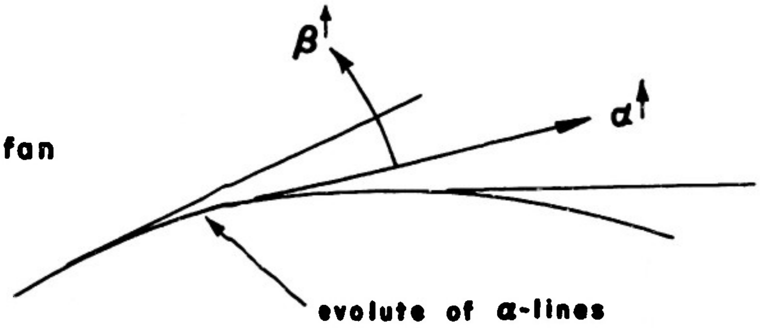


Fig. 8 Co-ordinate system used in investigation of stress and velocity equations

a) non-centered fan



b) centered fan

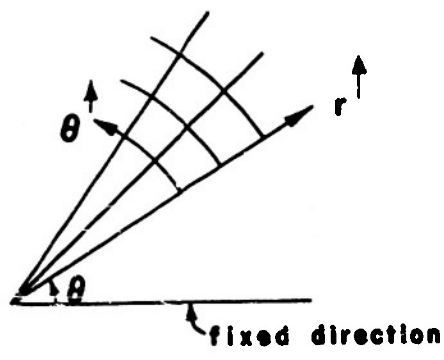


Fig. 9 Principal stress trajectories for plastic regime AB

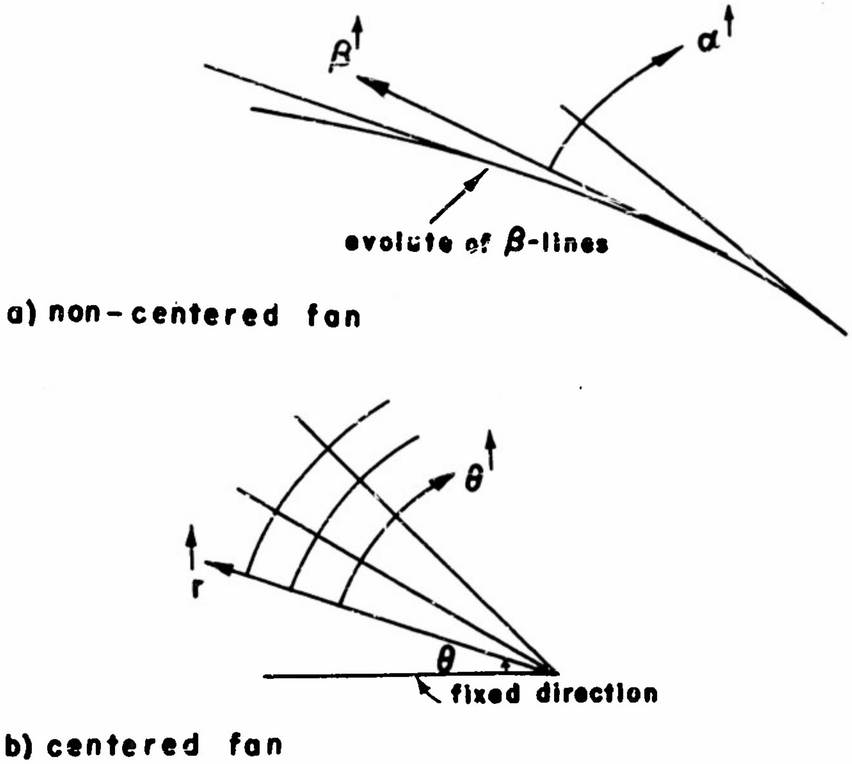


Fig. 10 Principal stress trajectories for plastic regime BC

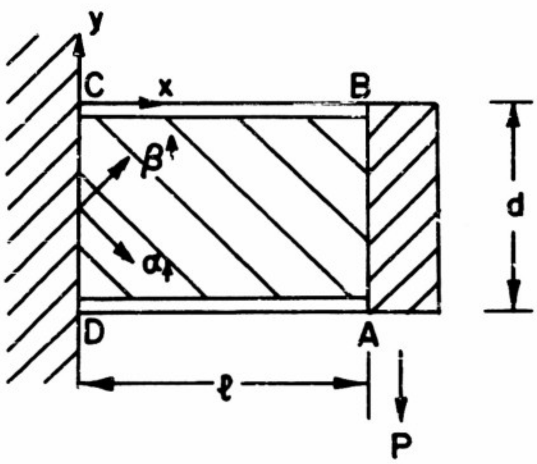


Fig. 11 Tension-field beam with no stiffeners